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The Effect of Increasing the Rate of Repetitions of Classical Reactions

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Using quantum theory operator method, we discuss the general reversible classical reactions $A_1 + A_2 + \cdots A_r \leftrightarrow B_1 + B_2 + \cdots + B_s$, where r and s are arbitrary natural positive numbers. We show that if either direction of the reaction is repeated a large number of times N in a finite total times T then in the limit of very large N, keeping T constant, one remains with the initial reacting particles only. We also show that if the reaction evolves through different possible paths of evolution, each of them beginning at the same side of the reaction, proceeds through different intermediate consecutive reactions and ends at the other side, then one may "realize" any such path by performing in a dense manner the set of reactions along it. the same results are also numerically demonstrated for the specific reversible reaction $A + B \leftrightarrow A + C$. We note that similar results have been shown to hold also in the quantum regime.

1. INTRODUCTION

The general reversible reactions $A_1 + A_2 + \cdots + A_r \leftrightarrow B_1 + B_2 + \cdots + B_s$, where r and s are two arbitrary positive natural numbers, have been studied by many authors (see, for example, (Ben-Avraham and Havlin, 2000), and references therein). These studies discuss, especially, the effects of the single reaction, or, in case it is repeated N times, the effect of these repetitions where the general total time T increases proportionally to N. We can, however, imagine a situation in which the *rate* of these repetitions increases and discuss the effect of this increase upon the reaction. Such an effect have been studied in Kac (1947) with respect to random walk and it was shown that when the rate of repeating this walk becomes very large one obtains a Brownian motion (Nelson, 1967). It has also been shown (Bar, 2001, 2003) with respect to the finite one-dimensional array of N imperfect traps (Ben-Avraham and Havlin, 2000; Smoluchowski, 1917) that as N becomes very large the survival probability (Ben-Avraham and Havlin, 2000) of the particles that pass through them tends to unity. We show in this work that if either direction of the mentioned reversible reaction is repeated a large number N

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of times in a finite total time T, then in the limit of very large N (where T is kept constant) one remains with the initial reacting particles only.

We use quantum theory methods and terminology as done by many authors that use quantum formalism for analyzing classical reactions (see for example, Mattis and Glasser, 1998; and annotated bibligraphy therein). The most suitable quantum method is the coherent state one (Glauber, 1963; Klauder and Sudarshan, 1968) which allows us to define simultaneously, as has been remarked in Swanson (1992), the expectation values of the conjugate variables Q and P. Thus, they may both have nonzero values in which case the entailed formalism resembles (Swanson, 1992) the classical one and so it may be used for discussing classical reactions.

The use of the coherent state formalism, together with second quantization methods, for classical systems have been extensively studied by Masao (1976). Since the described phenomena and, especially, the particles participating in the reactions are classical we represent them here by *real* coherent states. That is, we denote the reacting and product particles by the real coherent states $|z> = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{n=\infty} z^n/(n!)^{\frac{1}{2}}|n>$, where z is the real number $z = q + p/(2)^{\frac{1}{2}}$, q and p are two arbitrary real c numbers and $|n>$ are number representation eigenstates (Klauder and Sudarshan, 1968). The masses of the reacting and product particles are assigned, for convenience, the unity value. We note that although the mathematical entities and "operators" involved in this method do not conform, as will be shown, to the known quantum operator formalism, we, nevertheless, follow, except for the differences, the conventional definitions and methods of the last theory. The results obtained by applying the real coherent state formalism for classifical reactions are exactly the same as those previously obtained (Bar, 2000) by applying the complex coherent state methods for quantum reactions. Moreover, although the real coherent state formalism entails an evolution operator (see Eq. (3)), which is nonunitary and unbounded we, nevertheless, show that the results obtained are valid also for this kind of operator. That is, we obtain for the classical reactions the same results that were obtained (Bar, 2000) for the quantum analogues which are discussed by using complex coherent state formalism (Glauber, 1963; Klauder and Sudarshan, 1968), which involves unitary and bounded evolution operators. Needless to say that one may use the conventional complex coherent state formalism (Glauber, 1963; Klauder and Sudarshan, 1968) for discussing classical reactions.

In Section 2, we discuss the reversible reaction $A_1 + A_2 + \cdots + A_r \leftrightarrow B_1$ $B_2 + \cdots + B_s$ for the special cases of $r = s = 1$ and $r = s = 2$. That is, we study the reversible reactions $A \leftrightarrow B$ and $A_1 + A_2 \leftrightarrow B_1 + B_2$, and show that repeating either direction of each a large number N of times in a finite total time T results, in the limit of very large N, in a unity probability to remain with the initial reacting particles only. The generalization to any finite r and s follows.

We note that since we discuss the probability to remain with the initial reacting particles the product particles of such reactions are not relevant (as the reacting ones) to our discussion. In Section 3 we discuss the more general and natural case in which the product particles are relevant. That is, we assume that the particles of the ensemble interact at different places and times and that they begin from some given initial configuration of reactions and end at a final *different* one. We assume that there are large number of different paths of reactions that all begin at the given initial configuration and end at the final one and we calculate the probability that a specified system of reacting particles evolves along some prescribed path of them. We note that such paths of "states" for the diffusion controlled reactions have been discussed in (Masao, 1976; Mikhailov, 1981; Mikhailov and Yashin, 1985; Namiki, 1992) where use was made of quantum field theory methods (Mahan, 1993; Enz, 1992; Mattuck, 1976), including Wick's theorem (Mahan, 1993; Enz, 1992; Mattuck, 1976), to derive the classical Feynman diagrams and paths (Feynman, 1948; Feynman and Hibbs, 1965). These methods were also used in (Mikhailov, 1981; Mikhailov and Yashin, 1985) for chemical kinetics. We show that taking the limit of a very large number N of reactions along the prescribed path and performing them in a dense manner one obtains a unity value for the probability of evolution along that path.

In Section 4, we use a numerical model that has been used in Bar (2001) for showing the effect of dense measurement on classical systems. This is the model of two dimensional concentric billiard (Bar, 2001) that is used here to numerically simulate the reversible reaction $A + B \leftrightarrow A + C$. The two possible modes of reflections inside the billiard; either between the two concentric circles or between points of the outer circle, represent the two directions of the reaction. We note that nuclear and radioactive reactions are well simulated by billiards in which the stationary scattering circles represent the interactions between particles (see, for example (Bauer and Bertsch, 1990) in which a model of a rectangular billiard with a circle inside was used to discuss the decay law of classical systems). We show that the numerical billiard simulations conform to the analytical results of Sections 2 and 3.

2. THE REVERSIBLE REACTION $A_1 + A_2 + \cdots A_R \leftrightarrow B_1 + B_2 + \cdots + B_S$

We assume that we have some set of N identical particles so that the configuration in which the ith particle is located at $q_i (i = 1, 2, \ldots, N)$ is defined as a state of the system and denoted, in Dirac's notation, $|q_1, q_2, \ldots, q_N\rangle (|q^N\rangle)$ (for distinguished sets of particles we partition (Masao, 1976) the total N system to $N_1, N_2, \ldots,$ subsystems). Thus, when representing in the following classical particles by states we mean that they are elements of some configuration of the whole system. Following this terminology we may calculate the probability to find the set of particles in some definite state $|q^N\rangle$ as (Masao, 1976)

$$F^{(N)}(q_1, q_2, \ldots, q_N; t) = \sum_{\text{all permutations of } q_i} f^{(N)}(q_1, q_2, \ldots, q_N; t)$$

where $f^{(N)}(q_1, q_2, \ldots, q_N; t)$ is some normalized distribution function. To this probability one assign, as done in Masao, (1976), a "state" $|F(t)\rangle = \sum_{N=0}^{\infty} \int dQ^N F^{(N)}(q_1, q_2, \ldots, q_N; t)|q^N\rangle$, where $\int dQ^N = \frac{\int dq^N}{N!}$ (the division by $N!$ is necessary (Masao, 1976) so as not to overcount the state $|q^N >$ $N!$ times). Thus, the former probability to find the system in the state $|q^N\rangle$ may be written as (Masao, 1976) $F^{(N)}(q_1, q_2, \ldots, q_N; t) = \langle q^N | F^{(t)} \rangle$.

We discuss first the general reversible reaction $A_1 + A_2 + \ldots\ A_r \leftrightarrow B_1 + B_2 + \cdots + B_s$ for the specific case of $r = s = 1$, $A \leftrightarrow B$ where A and B are, as noted, represented by the two coherent states (Klauder and Sudarshan, 1968)

$$\begin{aligned} |z_A\rangle &= e^{-\frac{1}{2}|z_A|^2} \sum_{n=0}^{n=\infty} \frac{z_A^n}{(n!)^{\frac{1}{2}}} |n\rangle \\ |z_B\rangle &= e^{-\frac{1}{2}|z_B|^2} \sum_{n=0}^{n=\infty} \frac{z_B^n}{(n!)^{\frac{1}{2}}} |n\rangle \end{aligned} \tag{1}$$

Using the following general equation for any two operators X and Y (Klauder and Sudarshan, 1968)

$$e^Y X e^{-Y} = X + [Y, X] + \frac{1}{2!}[Y, [Y, X]] + \cdots$$

where $[Y, X]$ is the commutation $[Y, X] = YX - XY$, one obtains

$$U(q, p)(\alpha P + \beta Q)U^{-1}(q, p) = \alpha(P + p) + \beta(Q + q) \tag{2}$$

The α, β are arbitrary parameters, $U(q, p)$ and $U^{-1}(q, p)$ are given respectively by $U(q, p) = e^{pQ - qP}$, $U^{-1}(q, p) = U(-q, -p)$, and Q, P are the coherent state operators that satisfy $[Q_i, P_j] = \delta_{ij}$. That is, $U(q, p)$ translates the operators Q and P by q and p, respectively. Now, since the coherent states have been defined, as remarked, in terms of the number representation eigenstates (see Eq. (1)) we write the time evolution operator of the relevant states as e^{Nt}, where N is the number operator (Klauder and Sudarshan, 1968) (note that since we discuss in this paper real coherent states the evolution operator is real also).

$$N = a^{\dagger} a = \left(\frac{Q - P}{\sqrt{2}}\right)\left(\frac{Q + P}{\sqrt{2}}\right) = \frac{1}{2}(Q^2 - P^2 + 1)$$

N is defined analogously to the corresponding operator of the complex coherent state formalism (Glauber, 1963; Klauder and Sudarshan, 1968) but without the complex notation i in the middle expression. Note that the operator N is not positive definite and this to remind us, as remarked, that the real coherent state formalism discussed here does not conform to the conventional quantum operator process. Nevertheless, as remarked, the final results obtained here are exactly the same as those accepted in (Bar, 2000) from the quantum complex coherent state formalism. The commutation $[Q_i, P_j] = \delta_{ij}$ have been used at the right-hand side of the last equation. Applying the operator e^{Nt} on the coherent state $|z\rangle$ from Eq. (1), and taking the scalar product of the result with the conjugate state $\langle z|$ one obtain (using $\langle n|e^{Nt}|m\rangle = e^{nt}\delta_{nm}$, since in the number representation the operator N is diagonal)

$$\begin{aligned}\langle \grave{z}|e^{Nt}|z\rangle &= \exp\left(-\frac{1}{2}|z|^2 - \frac{1}{2}|\grave{z}|^2\right)\sum_{n=0}^{n=\infty}\frac{(\grave{z}e^t z)^n}{n!}\\ &= \exp\left(-\frac{1}{2}|z|^2 - \frac{1}{2}|\grave{z}|^2 + \grave{z}e^t z\right) = \langle \grave{z}|e^t z\rangle \\ &= \langle \grave{z}|(\cosh t + \sinh t)z\rangle = \langle \grave{q}, \grave{p}|q_t, p_t\rangle\end{aligned} \tag{3}$$

The last result is obtained by writing z in terms of q, p in which we have

$$q_t = q(\cosh t + \sinh t), \quad p_t = (\cosh t + \sinh t) \tag{4}$$

We, now, calculate, using Eq. (3), the probability $p(|q_A, p_A\rangle)$ to remain with the initial particle A after the reaction $A \to B$ where the particle B is represented by the coherent state $e^{Nt}|z_A\rangle$. This is given by

$$\begin{aligned}\langle z_A|e^{Nt}|z_A\rangle = \langle q_A, p_A|q_{A_t}, p_{A_t}\rangle &= \exp\Bigg(-\frac{1}{4}(q_A + p_A)^2\\ &\quad - \frac{1}{4}\left(q_{A_t} + p_{A_t}\right)^2\Bigg)\sum_{m,n=0}^{m,n=\infty}\frac{\left(q_A + p_A\right)^m\left(q_{A_t} + p_{A_t}\right)^n}{2^{\frac{m+n}{2}}(m!n!)^{\frac{1}{2}}}\langle m \mid n\rangle\\ &= \exp\left(-\frac{1}{4}(p_A + q_A)^2 - \frac{1}{4}\left(p_{A_t} + q_{A_t}\right)^2\right)\\ &\quad \times \sum_{n=0}^{n=\infty}\frac{(q_A + p_A)^n(q_{A_t} + p_{A_t})^n}{2^n n!}\\ &= \exp\Bigg(-\frac{1}{4}(p_A + q_A)^2 - \frac{1}{4}\left(p_{A_t} + q_{A_t}\right)^2\\ &\quad + \frac{1}{2}(q_A + p_A)\cdot\left(q_{A_t} + p_{A_t}\right)\Bigg)\\ &= \exp\Bigg(-\frac{1}{4}(p_A + q_A)^2 - \frac{1}{4}(p_A + q_A)^2(\cosh t + \sinh t)^2\end{aligned}$$

$$+\frac{1}{2}(q_A+p_A)^2\cdot(\cosh t+\sinh t)\Bigg)$$

$$=\exp\left(-\frac{1}{2}(p_A+q_A)^2\left(\frac{1}{2}+\frac{1}{2}(\cosh t+\sinh t)^2\right.\right.$$

$$\left.\left.-(\cosh t+\sinh t)\right)\right) \tag{5}$$

Note that since we discuss coherent states the interpretation (Klauder and Sudarshan, 1968) of the expression $\langle z_A|e^{Nt}|z_A\rangle$ is the probability to find the *mean* position and momentum of the coherent state $e^{Nt}|z_A\rangle$, which represents B, equal to those of z_A, which represents A, and this probability is equivalent in our discussion to remaining with the particle A. From Eq. (5) one obtains the probability to remain with the initial particle A after a single reaction $A \rightarrow B$. If it is repeated n times in a finite total time T one obtains (using $n=\frac{T}{\delta t}$, where δt is the duration of each reaction)

$$p^n(|q_A,p_A\rangle)=\exp\left(-\frac{T}{2\delta t}(p_A+q_A)^2\left(\frac{1}{2}+\frac{1}{2}(\cosh\delta t+\sinh\delta t)^2\right.\right.$$

$$\left.\left.-(\cosh\delta t+\sinh\delta t)\right)\right) \tag{6}$$

In the limit of very large n (very small δt) we expand the hyperbolic functions in a Taylor series and keep terms up to second order in δt. We obtain

$$p^n(|q_A,p_A\rangle)$$

$$=\exp\left(-\frac{T}{2\delta t}(p_A+q_A)^2\left(\frac{1}{2}+\frac{1}{2}(1+2\delta t^2+2\delta t)-\left(1+\frac{\delta t^2}{2}+\delta t\right)\right)\right)$$

$$=\exp\left(-\frac{T}{4\delta t}(p_A+q_A)^2\delta t^2\right)=\exp\left(-\frac{T}{4}(p_A+q_A)^2\delta t^2\right) \tag{7}$$

Thus, in the limit $n\rightarrow\infty(\delta t\rightarrow 0)$ we have

$$\lim_{n\rightarrow\infty}p^n(|q_A,p_A>)=\lim_{\delta t\rightarrow 0}\exp\left(-\frac{T}{4}(p_A+q_A)^2\delta t\right)=1 \tag{8}$$

Now, although we refer in the former equations to the direction $A\rightarrow B$ all our discussion remains valid also for the opposite one $B\rightarrow A$. That is, repeating either side of the reaction $A\leftrightarrow B$ a large number of times n in a finite total time T results, in the limit of very large n, in remaining (with probability 1) with the initial particles of the repeated reaction.

We, now, discuss the reversible reaction $A+B\leftrightarrow C+D$ in which we have two reacting particles. We continue to use the number evolution operator of N and take into account that the initial particles A and B interact. Thus, representing these particles as the coherent states $|q_A, p_A\rangle \cdot$ and $|q_B, p_B\rangle$ we write, for example, the left-hand side direction of the former reversible reaction $A + B \rightarrow C + D$ as

$$\exp((N_A + N_B + P_A P_B + Q_A Q_B)t)|q_A, p_A\rangle q_B, p_B\rangle = |q_C, p_C\rangle|q_D, p_D\rangle \tag{9}$$

where the terms $Q_A Q_B$ and $P_A P_B$ represent, as for the boson particles discussed in (Klauder and Sudarshan, 1968), the interaction of the particles A and B, and N_A, N_B are the number operators for them. Note that, as for the reaction $A \leftrightarrow B$ (see the discussion after Eqs. (4) and (5)), the operation of the evolution operator on the coherent state $|q_A, p_A\rangle|q_B, p_B\rangle$, which is now more complicated due to the interaction between A and B, is represented by $|q_C, p_C\rangle|q_D, p_D\rangle$. We calculate, now, the probability that the reaction $A + B \rightarrow C + D$ results in remaining with the initial particles A and B only (we denote this probability by $p(|q_B, p_B\rangle|q_A, p_A\rangle)$).

$$\begin{aligned} p(|q_B, p_B\rangle|q_A, p_A\rangle) = \langle q_B, p_B|\langle q_A, p_A|\exp((N_A + N_B + P_A P_B \\ + Q_A Q_B)t)|q_B, p_B\rangle q_A, p_A\rangle \end{aligned} \tag{10}$$

Using Eqs. (1), (3), and the following coherent states properties (Klauder and Sudarshan, 1968)

$\langle q, p|Q|q, p\rangle = q$, $\langle q, p|P|q, p\rangle = p$ (derived by using the operator U from Eq. (2) and the relation $N|0, 0\rangle = 0$) we obtain

$$\begin{aligned} p(|q_B, p_B\rangle|q_A, p_A\rangle) &= \exp((q_A q_B + p_A p_B)t) \\ &\quad \cdot\langle q_B, p_B|\langle q_A, p_A|q_{B_t}, p_{B_t}\rangle|q_{A_t}, p_{A_t}\rangle \\ &= \exp((q_A q_B + p_A p_B)t) \cdot \exp\bigg(-\frac{1}{4}(q_A + p_A)^2 \\ &\quad -\frac{1}{4}(q_B + p_B)^2 - \frac{1}{4}\left(q_{A_t} + p_{A_t}\right)^2 - \frac{1}{4}\left(q_{B_t} + p_{B_t}\right)^2\bigg) \\ &\quad \cdot \sum_{m,n=0}^{m,n=\infty} \frac{(q_A + p_A)^m \left(q_{A_t} + p_{A_t}\right)^n}{2^{\frac{m+n}{2}}(m!n!)^{\frac{1}{2}}} \langle m|n\rangle \\ &\quad \sum_{s,r=0}^{s,r=\infty} \frac{(q_A + p_A)^s \left(q_{A_t} + p_{A_t}\right)^r}{2^{\frac{s+r}{2}}(s!r!)^{\frac{1}{2}}} \cdot \langle s|r\rangle \\ &= \exp((q_A q_B + p_A p_B)t)\exp\bigg(-\frac{1}{4}(q_A + p_A)^2 \\ &\quad -\frac{1}{4}(q_B + p_B)^2 - \frac{1}{4}\left(q_{A_t} + p_{A_t}\right)^2 - \frac{1}{4}\left(q_{B_t} + p_{B_t}\right)^2\bigg) \end{aligned}$$

$$
\begin{aligned}
&\sum_{n=0}^{n=\infty} \frac{(q_A + p_A)^n \left(q_{A_t} + p_{A_t}\right)^n}{2^n n!} \\
&\cdot \sum_{r=0}^{r=\infty} \frac{(q_A + p_A)^r \left(q_{A_t} + p_{A_t}\right)^r}{2^r r!} \\
&= \exp((q_A q_B + p_A p_B)t) \exp\Bigg(-\frac{1}{4}(q_A + p_A)^2 \\
&\quad -\frac{1}{4}(q_B + p_B)^2 - \frac{1}{4}\left(q_{A_t} + p_{A_t}\right)^2 - \frac{1}{4}\left(q_{B_t} + p_{B_t}\right)^2 \\
&\quad + \frac{1}{2}(q_A + p_A)\left(q_{A_t} + p_{A_t}\right) + \frac{1}{2}(q_B + p_B)\left(q_{B_t} + p_{B_t}\right)\Bigg) \\
&= \exp\Bigg(\frac{1}{2}(\cosh\ t + \sinh\ t)((q_A + p_A)^2 + (q_B + p_B)^2) \\
&\quad -\frac{1}{4}(1 + (\cosh\ t + \sinh\ t)^2).((q_A + p_A)^2 \qquad (11) \\
&\quad + (q_B + p_A)^2) + (q_A q_B + p_A p_B)t \Bigg)
\end{aligned}
$$

This is the probability to remain with the original particles A and B after one reaction. Repeating it a large number of times n in a finite total time T, where $n = \frac{T}{\delta t}$ (δt is the time duration of one reaction) one obtains for the probability to remain with A and B.

$$
\begin{aligned}
P^n(|q_B, p_B\rangle |q_A, p_A\rangle) = \exp\Bigg(\frac{T}{\delta t}\Bigg(&(q_A q_B + p_A p_B)\delta t \\
&+ \Bigg(\frac{1}{2}(\cosh \delta t + \sinh \delta t) \\
&- \frac{1}{4}(1 + (\cosh \delta\ t + \sinh \delta\ t)^2)\Bigg) \\
&. ((q_A + p_A)^2 + (q_B + p_B)^2)\Bigg)\Bigg)
\end{aligned}
\tag{12}
$$

In the limit of very large n we expand the hyperbolic functions in a taylor series and retain terms up to second order in δt. Thus,

$$
\begin{aligned}
P^n(|q_B, p_B\rangle |q_A, p_A\rangle) = \exp\Bigg(\frac{T}{\delta t}\Bigg(&(q_A q_B + p_A p_B)\delta t + \Bigg(\frac{1}{2}\left(1 + \frac{\delta t^2}{2} + \delta t\right) \\
&- \frac{1}{4}(2 + 2\delta t^2 + 2\delta t)\Bigg)((q_A + p_A)^2 + (q_B + p_B)^2)\Bigg)\Bigg)
\end{aligned}
$$

$$= \exp\left(T\left((q_A q_B + p_A p_B) - \frac{\delta t}{4}((q_A + p_A)^2 + (q_B + p_B)^2)\right)\right) \tag{13}$$

Taking the limit of $n \to \infty$ ($\delta t \to 0$) we obtain

$$\lim_{n\to\infty} P^n(|q_B, p_B\rangle|q_A, p_A\rangle) = \exp(T(q_A q_B + p_A p_B)) \tag{14}$$

The last probability tends to unity when the c-numbers of either A or B (or both) are zeroes, that is, when A or B (or A and B) are in their ground states (in which case they are represented by only the first term of the sums in Eq. (1)). Needless to say that all the former discussion remains valid also for the opposite direction $A + C \to A + B$. Thus, we conclude that when either direction of the reversible reaction $A + B \leftrightarrow C + D$ is repeated a large number of times n in a finite total time and when at least one of the reacting particles was in the ground state so that its c-numbers are zeroes one obtains, in the limit of very large n, a result as if the repeated reaction did not occur at all.

It can be shown that the general reversible reaction $A_1 + A_2 + \cdots + A_r \leftrightarrow B_1 + B_2 + \cdots + B_s$, where r, s are any two positive natural numbers, also results in a similar outcome if at least one of the reacting particles has zero c numbers. We note that the last condition is not necessary when we begin with only one reacting particle as we see from the reaction $A \leftrightarrow B$.

3. THE PROBABILITY TO FIND GIVEN FINAL CONFIGURATION DIFFERENT FROM THE INITIAL ONE.

We now discuss the more general and natural case in which we have an ensemble of particles and we calculate the probability to find at the time t a subsytem of this ensemble at some given configuration if at the initial time t_0 it was at another prescribed configuration. We assume that the corresponding time difference $(t - t_0)$ is not infinitesimal and that during this time the subsystem has undergones a series of reactions. The passage from some reaction at some intermediate time t_i to the neighbouring one at the time $(t_i + \delta t)$ is governed by the correlation between the corresponding resulting configurations of the subsystem at these times. Thus, restricting, for the moment, our attention to the case in which a particle in the subsystem that was at the time t_0 in the state A and at the time $(t_0 + \delta t)$ in B we can write the relevant correlation (Klauder and Sudarshan, 1968) between these two states as

$$\tau\big(A, B; t_{0_A}, (t_0 + \delta t)_B\big) = \langle q_{A_{t_0}}, p_{A_{t_0}} | q_{B_{t_0+\delta t}}, p_{B_{t_0+\delta t}}\rangle \tag{15}$$

where $|q_{A_{t_0}}, p_{A_{t_0}}\rangle$ and $|q_{B_{t_0+\delta t}}, p_{B_{t_0+\delta t}}\rangle$ are the coherent states that represent the particles A and B at the times t_0 and $t_0 + \delta t$, respectively (see Eqs. (3) and (4)) and the angular brackets denote an ensemble average over all the particles of it. We note that if $A = B$, τ measures (Klauder and Sudarshan, 1968) the autocorrelation of either the particle A or B, and when $A \neq B$, τ is the crosscorelation (Klauder and Sudarshan, 1968) of the two particles. It can be shown, using Eqs. (1) and (15) that the following relation

$$\left|\tau\left(A, B; t_{0_A}, (t_0 + \delta t)_B\right)\right|^2 = \tau\left(A, A; t_{0_A}; t_{0_A}\right)\tau(B, B; (t_0 + \delta t)_B, (t_0 + \delta t)_B) \tag{16}$$

is valid. That is, the modulus of the crosscorrelation of the particles A and B at the times t_0 and $(t_0 + \delta t)$ equals the product of the autocorrelation of the particle A at the time t_0 by that of B at the time $(t_0 + \delta t)$. The last relation is interpreted (Klauder and Sudarshan, 1968) as the probability density for the occurrence of the reaction $A \rightarrow B$ at the time $(t_0 + \delta t)$. That is, given that the system was in "state" A at the time t_0, the probability to find it at the time $(t_0 + \delta t)$ in "state" B is given by Eq. (16). We can generalize to the joint probability density for the occurrence of n different reactions between the initial and final times t_0 and t, where each involves two different particles and is of the kind $A_i \rightarrow A_{i+1}$. That is, each reaction is composed of two parts; the first one is that in which a particle of the subsystem is observed at the time t_i to be in state A_i, and the second that in which it is observed at the time $t_i + \delta t$ to be in the state A_{i+1}. Thus, the total time interval $(t - t_0)$ is partitiond into $2n$ subintervals during which the n reactions occur. The total correlation is

$$\begin{aligned}
&|\tau(A_1, A_2, \ldots, A_{2n}; t_0, t_0 + \delta t, \ldots, t)|^2 \\
&= \tau(A_1, A_1; t_0, t_0) \cdot \tau(A_2, A_2; t_0 + \delta t, t_0 + \delta t) \cdots \tau(A_{2n}, A_{2n}; t, t) \\
&= \prod_{k=0}^{k=2n-1} \tau(A_{k+1}, A_{k+1}; t_0 + k\delta t, t_0 + k\delta t) = \prod_{k=0}^{k=n-1} \tau(A_{2k+1}, A_{2k+1}; t_0 \\
&\quad +2k\delta t, t_0 + 2k\delta t) \cdot \tau(A_{2k+2}, A_{2k+2};\ t_0 + (2k+1)\delta t, t_0 + (2k+1)\delta t) \\
&= \prod_{k=0}^{k=n-1} |\tau(A_{2k+1}, A_{2k+2}; t_0 + 2k\delta t,\ t_0 + (2k+1)\delta t)|^2
\end{aligned} \tag{17}$$

The last result was obtained by using Eq. (16). By the notation A_{2n} we mean, as remarked, that there are n separate reactions each of which involves two states (and not $2n$ different particles). Now, we show in the previous section, for either direction of the reversible reaction $A \leftrightarrow B$, that the probability to remain in the initial state A(or B) tends to unity in the limit of a very large number of repetitions, in a finite total time, of $A \rightarrow B$ (or $B \rightarrow A$) which amounts to performing each such reaction in an infinitesimal time δt. That is, in this limit of vanishing δt each factor of the last product in Eq. (17), which is the probability for the reaction $A_i \rightarrow A_{i+1}$, tends to unity and with it the joint probability for the occurrence of the n reactions. Thus, the specific prescribed path of reactions is followed with a probability of unity.

From the last discussion we may obtain the joint probability density for the case in which some of the n intermediate reactions are of the more general kind $A_1 + A_2 + \cdots + A_r \rightarrow B_1 + B_2 + \cdots + B_r$, where r is an arbitrary natural positive number. That is, at some of the $2n$ times there may occur, in a simultaneous manner, r different reactions at r different places each of the kind $A \rightarrow B$. Thus, we assume that r particles in the subsystem that were at the time $t_o + i\delta t$, in the states $A_j (j = 1, 3, 5, \ldots, 2r - 1)$. were observed at the time $(t_0 + (i + 1)\delta t)$ to be in the states $A_{j+1} (j + 1 = 2, 4, 6, \ldots, 2r)$. We assume that at each of the other intermediate times there occurs only one single reaction $A_i \rightarrow A_{i+1}$. Thus, there are $(n + r - 1)$ reactions each of them occurs between two particles. In this case the corresponding total coherence among all these reactions is

$$\tau_{\text{total}} = \tau(A_1, A_2, \ldots A_{i+1}, A_{i+2}, \ldots, A_{i+2r}, \ldots, A_{(2n+2r-2)}; t_0, t_0 + \delta t, \\ \ldots, \underbrace{t_0 + i\delta t, t_0 + (i + 1)\delta t, \ldots t_0 + i\delta t, t_0 + (i + 1)\delta t}_{r}, \ldots, t)$$

where the underbrace denotes that the r particles observed at the time $(t_0 + i\delta t)$ as A_j were seen to be at the time $t_0 + (i + 1)\delta t$ as $A_{j+1} (j = 1, 3, \cdots, 2r - 1)$. Again the notation $A_{(2n+2r-2)}$ means that we have $(n + r - 1)$ reactions each involving, as remarked, two states. Using Eqs. (16) and (17) and the former equation for τ_{total} we find that the joint probability to find at the time t the relevant subsystem at the given final configuration after the occurence of these $(n + r - 1)$ reactions is

$$|\tau_{\text{total}}|^2 = \tau(A_1, A_1; t_0, t_0)\tau(A_2, A_2; t_0 + \delta t, t_0 + \delta t) \ldots \tau(A_{i+1}, A_{i+1}; t_0 \\ + i\delta t, t_0 + i\delta t) \cdot \tau(A_{i+2}, A_{i+2}; t_0 + (i + 1)\delta t, t_0 + (i + 1)\delta t) \\ \ldots \tau(A_{i+2r-1}, A_{i+2r-1}; t_0 + i\delta t, t_0 + i\delta t) \cdot \tau(A_{i+2r}, A_{i+2r}; t_0 \\ + (i + 1)\delta t, t_0 + (i + 1)\delta t) \ldots \tau(A_{(2n+2r-2)}, A_{(2n+2r-2)}, t_0 + 2n\delta t, t_0 \\ + 2n\delta t) = \prod_{k=0}^{k=n-1} \tau(A_{2k+1}, A_{2k+1}; t_0 + 2k\delta t, t_0 + 2k\delta t) \\ \times \tau(A_{2k+2}, A_{2k+2}; t_0 + (2k + 1)\delta t, t_0 + (2k + 1)\delta t) \cdot \\ \prod_{j=1}^{r-1} \tau(A_{i+j}, A_{i+j}; t_0 + i\delta t, t_0 + i\delta t) \cdot \tau(A_{i+j+1}, A_{i+j+1}; t_0$$

$$+(i+1)\delta t, t_0+(i+1)\delta t) = \prod_{k=0}^{k=n-1} |\tau(A_{2k+1}, A_{2k+2}; t_0+2k\delta t, t_0$$

$$+(2k+1)\delta t)|^2 \prod_{j=1}^{j=r-1} |\tau(A_{i+j}, A_{i+j+1}; t_0+i\delta t, t_0+(i+1)\delta t|^2 \quad (18)$$

The first product of the last result is the same as that of Eq. (17) and the second takes account of the $r-1$ simultaneous reactions at the time $(t_0+(i+1)\delta t)$ (the first product involves also one of the r simultaneous reactions at the time $(t_0+(i+1)\delta t)$). Each of the reactions in both products is of the kind $A \rightarrow B$ which was shown in the former section (see also the discussion after Eq. (17)) to have a unity probability for remaining with the initial particle A in the limit in which the time duration of the reaction becomes infinitesimal. That is, in this limit in which the time alloted for each reaction $A \rightarrow B$ becomes very small each factor of each product of Eq. (18), and with it the whole expression, tends to unity. If any particle A of the sybsystem does not react with any other particle at some of the $2n$ intermediate times then we may denote its no-reaction at these times as $A \rightarrow A$ and the probability for it to remain in the initial state (which is the same as the final one) is obviously unity.

Thus, we see that the probability to find at the time t the given ensemble of particles following a given path of reactions (from a large number of possible paths) tends to unity if the relevant reactions are performed in a dense manner. That is, each occuring in an infinitesimal time δt.

4. BILLIARD SIMULATION OF THE REVERSIBLE REACTION $A + B \leftrightarrow A + C$

We, now, show that the results described in the previous sections have also strong numerical support. This is demonstrated for the reversible reaction $A+B \leftrightarrow A+C$, which is simulated here by using the two-dimensional circular billiard (Bar, 2001) which is composed of two concentric circles. We assume that initialy we have a large ensemble of identical point particles each of them is the component A of the reaction $A+B \leftrightarrow A+C$. All of these particles are entered, one at a time, into the billiard in which they are elastically reflected by the two concentric circles. That is, the angles before and after each reflection are equal. We assume that on the outer circle there is a narrow hole through which the particles A leave the billiard. Once a particle is ejected out a new one is entered and reflected inside the billiard untill it leaves and so for all the particles of the ensemble. There are two different possible kinds of motion for each point particle A before leaving the billiard; either it is reflected between the two concentric circles or, when the angle of reflection is large, reflected by the outer circle only without touching the inner one. Now, since both motions are elastic each particle A, once it begins its reflection in either kind of motion, continues to move only in this kind untill it leaves through the narrow opening. The component B of the reaction denotes the outer larger circle, and the component C denotes both circles. That is, the left hand side $A + B$ of the reaction signifies that the point particle A moves inside the billiard and is reflected by the outer circle only, whereas, the right-hand side $A + C$ denotes the second kind of motion in which the point balls A are reflected between the two circles. We call these two kinds of paths "states" (Bar, 2001), so that the path that touches both circles is "state" 1 and the one that touches the outer circle only is "state" 2. This billiard model was studied in (Bar, 2001) as an example of a classical system that behaves the same way quantum systems do when exposed to a large number of repetitions, in a finite total time, of the same experiment (Misra and Sudarshan, 1977; Giulini *et al.*, 1996; Pascazio and Namiki, 1994; Cook, 1988; Harris and Stodolsky, 1981; Bixon, 1982; Itano *et al.*, 1990; Kofman and Kurizki, 1996; Simonius, 1978; Aharonov and Vardi, 1980; Facchi *et al.*, 1999). The concentric billiard is, schematically, represented in Fig. 1.

Now, since in such a system we can not follow the parth of each particle and can not differentiate between the two kinds of motion we have to consider, as done for the nuclear and radioactive processes (Bauer and Bertsch, 1990), the activities of these particles in either path. That is, the rate at which the entire

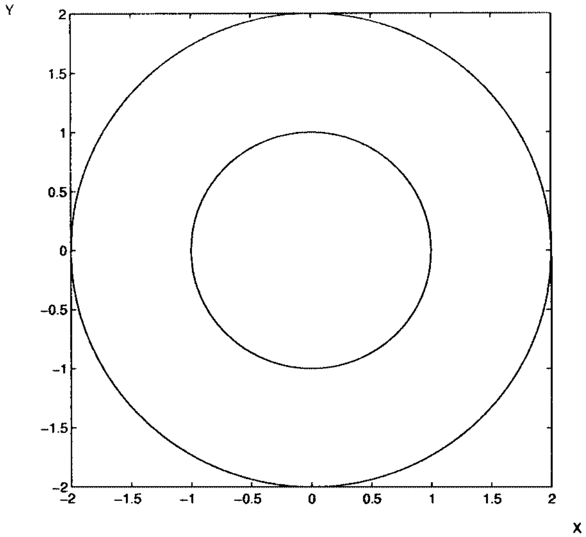

Fig. 1. A schematic representation of the concentric circular billiard that simulates the reversible reaction $A + B \leftrightarrow C + D$.

ensemble of particles, being at either state, leaves the billiard. We assume for the activity discussed here, as is assumed (Bauer and Bertsch, 1990) for the nuclear and radioactive's activities, that each particle A enjoys arbitrary initial conditions, so in the following numerical simulations we assume that it may begin its journey inside the billiard at either "state" which is determined randomly using a random number generator.

We want to show, numerically, that if either side of this reversible reaction $A + B \leftrightarrow A + C$ is repeated a large number of times N in a finite total time T, then, in the limit of very large N, the activity obtained is the same as the "natural activity" (Bar, 2001) that results when no such repetitions are performed (Bar, 2001). For that matter, we take into account that the reversible reactions that occur in nature have either equal or different rates for the two directions of the reactions and that the total activity of the ensemble of particles depends critically upon these rates (Bauer and Bertsch, 1990). If, for example, we consider the equal rate case then we have to discuss the rate of evacuation of the billiard when each particle is allowed, after a prefixed number of reflections in either state, to pass, if it is still in the billiard, to the other one. This activity is shown by the solid curve in Fig. 2 in which the ordinate axis denotes the number of particles A that leave the billiard in prescribed time intervals binned in units of 60 (Bauer and Bertsch, 1990). We

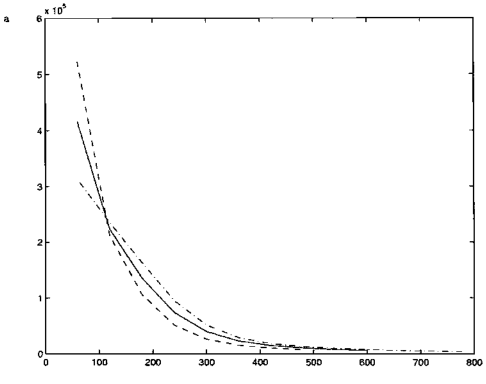

Fig. 2. The dashed curve shows the activity obtained when all the 10^6 particles of the ensemble are allowed to be only in "state" 2, in which they are reflected between points on the outer circle only. The dashdot curve is the activity when all these particles are allowed to move only in "state" 1 (between the two circles). For the values assigned here to the outer and inner circles (6 and 3) the dashed (dashdot) curve is the maximum (minimum) activity. The solid curve shows the activity obtained when the particles in either state pass to the other after every 1100 reflections. The x axis specifies time binned in units of 60.

assume (Bauer and Bertsch, 1990) that each point particle A in either state moves with the same speed of 3, and the hole through which they leave has a width of 0.15. We denote the outer and inner radii of the billiard by r_1 and r_2, respectively, and assign them the values of $r_1 = 6$ and $r_2 = 3$. The initial number of the particles A was 10^6, and each one of them passes from one "state" to the other, if it did not leave the billiard through the hole, after every 1100 consecutive reflections. We note that this rate of one passage for every 1100 reflections is typical and common for these kinds of billiard simulations (Bauer and Bertsch, 1990; Gutkin, 1986; Hobson, 1975).

The natural activity is obtained, as remarked, when the entire ensemble of 10^6 particles A enter, one at a time, to the billiard at the same "state" and remain all the time in this "state" without passing to the other until they leave the billiard. The dashed curve in Fig. 2 shows this natural activity when all the particles A are in "state" 2 in which they are reflected only between points of the outer circle untill they leave the billiard. The dash-dot curve shows the activity when all the particles A are in "state" 1 in which they are reflected only between the two circles. It has been found that for the values assigned here to the radii of the outer and inner circles (6 and 3) the activity of "state" 2 shown by the dashed curve is the maximum available and that of state 1 shown by the dashdot curve is the minimum. The large difference between the two activities has its source in the range of the allowed angles of reflections which is much larger in state 2 than in state 1. This is because the minimum trajectory between two neighbouring reflections in state 2, where the particles A are reflected between points of the outer circle only, may be infinitesimal compared to the corresponding trajectory in state 1 which is (we denote the trajectories between neighboring reflections in states 1 and 2 by d_1 and d_2, respectively) $d_{1\,\min} = r_1 - r_2$. For the values assigned here to the radii r_1 and r_2 of the two concentric circles ($r_1 = 6$ and $r_2 = 3$) $d_{1\,\min} = 3$. We note that the maximum trajectory between two neighbouring reflections in state 2 is equal to the corresponding one in state 1, that is

$$d_{1\,\max} = d_{2\,\max} = \sqrt{r_1^2 + r_2^2}$$

Thus, the particles in A have many more possibilities to be reflected to the hole and leave the billiard in state 2 than in state 1 and, accordingly, their activity is much larger. The solid curve in Fig. 2 is, as remarked, the activity obtained when the particles A are transferred between the two states at the rate of one passage for every 1100 reflections and so, as expected, its activity is between the two other activities shown in Fig. 2.

We numerically interfere with the rate of the systematic passage of the point particles A between the two states such that this rate is accelerated. We refer in the following not only to a specific passage between the two "states" but also gives the results in parentheses for the opposite passage. Thus, it is found that the activity of the entire ensemble is directly (inversely) proportional to the rate of the passage from state 1 (2) to state 2 (1) when the opposite passage from state 2 (1) to state 1 (2) remains at the rate of one for every 1100 reflections. Thus, we have found that when the particles in state 1 (2) are transferred to state 2 (1) at the maximum rate of one passage for each single reflection and the particles in state 2(1) are passed to the state 1(2) at the rate of one for every 1100 reflections then the activity of the particles A is maximal (minimal). But as we have remarked the maximal (minimal) activity is obtained only when each particle of the entire ensemble is *always* in state 2(1). In other words, as we have remarked, a very large number of repetitions of the left (right) direction $A + B \rightarrow A + C(A + C \rightarrow A + B)$ of the reaction where the right (left) directon $A + C \rightarrow A + B(A + B \rightarrow A + C)$ occurs every 1100 reflections, yields a result as if the densely repeated reaction never happened and the activity obtained is the natural one in which no external repetition is present. The dashed curve in Fig. 3, which is the same as the dashed one of Fig. 2, shows the activity obtained when all the 10^6 particles A of the ensemble are allowed to move only in state 2 until they leave the billiard. The solid curve is the activity obtained when the reaction $A + C \rightarrow A + B$ is repeated

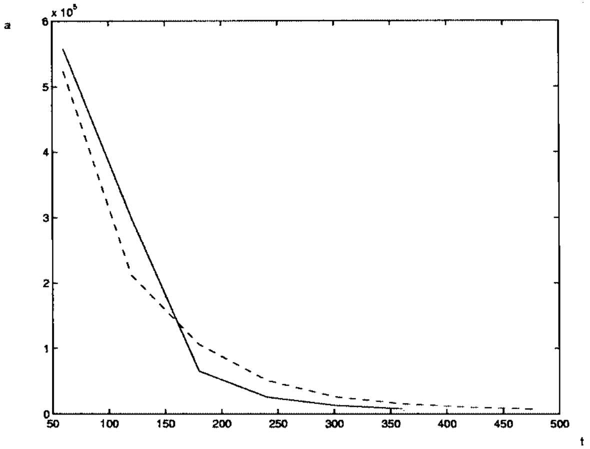

Fig. 3. The dashed curve, which is the same as the dashed curve of Fig. 2 (they look slightly different since the abcissa axes of these figures are different), shows the activity obtained when all the 10^6 particles A of the ensemble are numerically constrained to be only in "state" 2 until they evacuate the billiard. State 1 is not allowed for them. The solid curve is the activity obtained when each particle in state 1 is passed to "state" 2 after each single reflection, whereas those in "state" 2 pass to the opposite one only after every 1100 reflections. As for Fig. 2 the abcissa axis denotes time binned in units of 60. Note the similarity between the two curves.

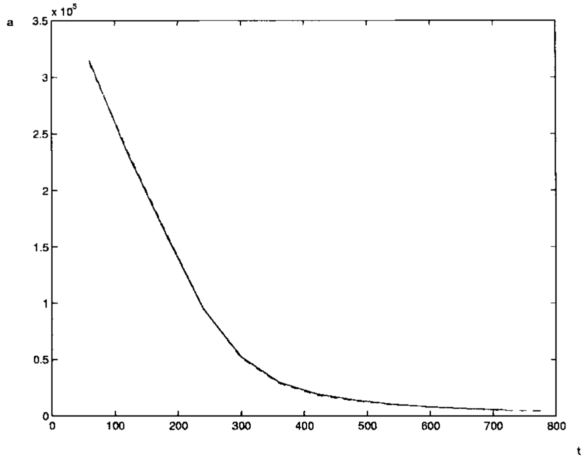

Fig. 4. The apparently one curve shown in the figure is actually two curves one dashed and the other solid. The dashed curve, which is the same as the dashdot curve of Fig. 2, shows the activity obtained when all the 10^6 particles A of the ensemble are numerically constrained to be only in "state" 1 until they evacuate the billiard. State 2 is not allowed for them. The solid curve is the activity obtained when each particle in state 2 is passed to "state" 1 after each single reflection, whereas those in "state" 1 pass to the opposite one only after every 1100 reflectons. As for Fig. 2 the abcissa axis denotes time binned in units of 60. Note that the two curves are almost identical (the dashed curve has a longer tail (for large t) than the solid one).

after each single reflection and the opposite one $A + B \rightarrow A + C$ after every 1100 reflections. It is seen that these curves are similar to each other. That is, we realize, in accordance with the analytical results of the former sections, that a large number of repetitions of the reaction yields a result that characterizes the activity obtained in the absence of such repetitions.

This is seen, in a much more clear way, in Fig. 4 for the other direction $A + B \rightarrow A + C$ of the reaction. The apparent single graph of the figure is actually composed of two curves; one solid and the other dashed. The solid curve shows the activity obtained when the reaction $A + B \rightarrow A + C$ is repeated after each single reflection and the opposite one $A + C \rightarrow A + B$ after every 1100 reflections. The dashed curve, which is identical to the dash-dot one from Fig. 2 is the activity obtained when all the particles A of the ensemble are constrained to move only in state 1 until they leave the billiard. Note that the two curves are almost the same except for the longer tail of the dashed curve.

From both Figs. 3 and 4, we realize that the large number of repetitions of either direction of the reversible reaction $A + B \leftrightarrow A + C$ has the effect as if it has not been performed at all and the actual activity obtained is that of the natural one that does not involve any repetitions.

We note that as the analytical results are obtained in the limit of the *largest* number (actually infinite) of repetitions so the similar numerical results are obtained in the limit of the largest number of repetitions of the reaction. That is, of numerically repeating it after each single reflection. In other words, a mere high rate (which is not the maximal) of one side of the reaction is not enough to produce the results shown in Figs. 3 and 4. This is clearly shown in Fig. 5 the solid curve of which shows the activity obtained when each particle in "state" 1 is passed to "state" 2 after every *two* neighboring reflections whereas those of "state" 2 are passed (one at a time) after every 1100 reflections. Note that the solid curve in Fig. 3 shows the activity obtained when the particles in "state" 1 are passed to "state" 2 after each reflection and those of 2 passed to 1 after every 1100 reflections. That is, although the two high rates represented by the two solid curves in Figs. 3 and 5 are almost the same nevertheless the resulting activities, contrary

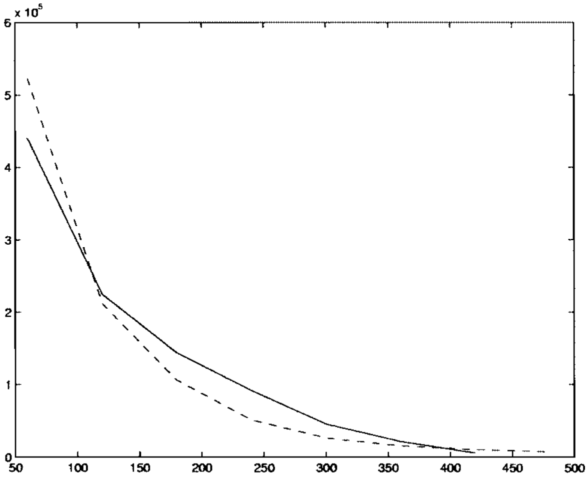

Fig. 5. The dashed curve, which is the same as the dashed curve of Fig. 3, shows the activity obtained when all the 10^6 particles A of the ensemble are moving inside the billiard only in "state" 2 until they are evacuated outside. State 1 is not allowed for them. The solid curve is the activity obtained when each particle in state 1 is passed to "state" 2 after every two reflection, whereas those in "state" 2 pass to the opposite one only after every 1100 reflections. Note that although the two solid curves of Fig, 3 and 5 differ slightly in their reaction rates (of one for each reflection in Fig. 3 and of one for each two reflections in Fig. 5) the activities are very different (see text). As for all the former figures the abcissa axis denotes time binned in units of 60.

to what one may expect, are very different. That is, that of Fig. 3 is much higher than that of Fig. 5 as may be seen from the solid curve that begins at $t = 60$ (note that our abcissa axis is binned in units of 60) from the high value of $5.65 \cdot 10^5$ and ends at $t = 360$. The corresponding solid curve of Fig. 5 begins at $t = 60$ at the much smaller value of $4.45 \cdot 10^5$ and ends at the later time of $t = 420$. That is, by only increasing the rate of repeating the same reaction from one for every two reflections to one for each reflection results in an additional 120, 000 particles that leave the billiard already at the first binned time unit. The two dashed curves of Figs. 3 and 5 are identical and denote the same activity obtained when all the 10^6 particles A of the ensemble are numerically constrained to be only in "state" 2 until they evacuate the billiard.

Thus, as remarked, the important factor that causes a result of maximum activity is the highest possible rate and not merely a large ratio between the higher and slower frequencies. This is in accord with the analytical results of Sections 2 and 3, which are obtained for the largest rate (actually infinite) of repeating any direction of the general reversible reaction $A_1 + A_2 + \cdots + A_r \leftrightarrow B_1 + B_2 + \cdots B_s$, where r and s are any two arbitrary natural positive numbers. In this limit we find, as found from the billiard simulation, that one remains, with a unit probability, with the initial reacting particles as if the repeated reaction did not occur at all.

All the former simulations were done when the outer and inner circles radii were 6 and 3, respectively. We note that we obtain similar numerical results for all other assigned values of r_1 and r_2 up to the extreme limits of $r_1 >> r_2$ and $r_1 \approx r_2$ provided we always have $r_1 > r_2$.

We may explain these results along the same line of explanation used to interpret the similar results obtained in the quantum regime. It have been established theoretically (Misra and Sudarshan, 1977; Simonius, 1978; Aharonov and Vardi, 1980; Pascazio and Namiki, 1994; Facchi *et al.*, 1999) and experimentally (Itano *et al.*, 1990; Kofman and Kurizki, 1996; Giulini *et al.*, 1996) that taking some quantum system which may reduce by experimenting on it to any of its relevant eigenstates and repeat this experiment a large number N of times in a finite total time T results, in the limit of very large N (keeping T constant), in preserving the initial state of the system. This phenomenon is the Zeno effect (Misra and Sudarshan, 1977; Simonius, 1978; Aharonov and Vardi, 1980; Pascazio and Namiki, 1994; Facchi *et al.*, 1999). The similar results obtained theoretically in Section 2 suggest that this effect may be effective also in the classical reactions. That is, repeating these reactions a large number of times, in a finite total time, may results in remaining with the initial reacting particles (initial "state"). Moreover, it have been shown (Aharonov and Vardi, 1980; Facchi *et al.*, 1999) that if any quantum system evolves in a number of different possible paths of states, each of which begins at the same given initial state and end at another given common final one, then it is possible to realize any such path by making dense measurement (in a finite total time) along it. That is, by performing in a dense manner the set of experiments that reduce the system to the specific states that constitute the relevant path. Similar results were obtained in Section 3, in which we show that the joint probability density for the occurence of n given different reactions between the initial and final times t_0 and t tends to unity in the limit of densely performing these reactions (when $n \rightarrow \infty$).

The same results were obtained also in the billiard simulations from which we realize that repeating a large number of times any direction of the reversible reaction $A + B \leftrightarrow A + C$ have, in the limit of numerically repeating it after each single reflection, the effect as if it has never occured. This is what one obtains in the Zeno effect (Misra and Sudarshan, 1977; Simonius, 1978; Aharonov and Vardi, 1980; Pascazio and Namiki, 1994; Facchi *et al.*, 1999; Itano *et al.*, 1990) where the system is preserved in the initial state in spite of the very large number of measurements.

CONCLUDING REMARKS

We have discussed the classical reactions using quantum theory methods in which particles are represented by coherent state functions and the product of these states with their conjugates is interpreted as probability (Masao, 1976). It is shown that if either direction of the general reaction $A_1 + A_2 + \cdots + A_r \leftrightarrow B_1 + B_2 + \cdots B_s$, where r and s are any two natural positive numbers, is repeated a large number N of times in a finite total time then in the limit of a very large N one remains, with a unity probability, with the initial particles only. In this context, we differentiate between the case in which there were more than one initial reacting particle and the case in which there is only one such particle. In the first case the mentioned unity probability is obtained if at least one of the initial reacting particles have zero values for its c-numbers q and p that denote its coherent state representation, whereas, in the second case this condition is not needed. Moreover, it has been shown that any prescribed evolution (from a large number of possible ones) through a sequence of specific reactions may be realized with a unit probability by densely performing these reactions. This effect that results from increasing the rate of the reaction have been demonstrated also through numerical simulation for the reversible reaction $A + B \leftrightarrow A + C$. We use for that purpose the two-dimensional concentric billiard in which the two modes of possible reflections represent the two sides of the reaction. It has been shown that by repeating either side of the reaction a large number N of times (whereas the other side was repeated with a much lower rate) we obtain, in the limit of the largest rate of repetition, a result as if the fastly repeated reaction were not performed at all.

The obtained results conform to the Zeno effect which is considered in Simonius (1978) to hold also in classical and macroscopic phenomena. In this effect (Misra and Sudarshan, 1977; Simonius, 1978; Pascazio and Namiki, 1994; Itano *et al.*, 1990) the very large number of repeating the same experiment, in a finite total time, results in preserving the system in the state it was before initiating these repetitions. Moreover, it has been shown in Aharonov and Vardi (1980) and Facchi *et al.*(1999) that this effect can be generalized to a whole path of states in which the final state is different from the initial one. That is, the mechanism of dense measurement caused the "realization" of this specific path from a very large number of different ones. This result was obtained by calculating the joint probability of n different reactions, each of which occurs at its specific place and time, where we see that densely performing these reactions causes the joint probability for the occurence of the n reactions (where $n \to \infty$) to tend to the unity value.

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